Strategyproofness and Proportionality in Party-Approval Multiwinner Elections

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Abstract

In party-approval multiwinner elections the goal is to allocate the seats of a fixed-size committee to parties based on the approval ballots of the voters over the parties. In particular, each voter can approve multiple parties and each party can be assigned multiple seats. Two central requirements in this setting are proportional representation and strategyproofness. Intuitively, proportional representation requires that every sufficiently large group of voters with similar preferences is represented in the committee. Strategyproofness demands that no voter can benefit by misreporting her true preferences. We show that these two axioms are incompatible for anonymous party-approval multiwinner voting rules, thus proving a far-reaching impossibility theorem. The proof of this result is obtained by formulating the problem in propositional logic and then letting a SAT solver show that the formula is unsatisfiable. Additionally, we demonstrate how to circumvent this impossibility by considering a weakening of strategyproofness which requires that only voters who do not approve any elected party cannot manipulate. While most common voting rules fail even this weak notion of strategyproofness, we characterize Chamberlin–Courant approval voting within the class of Thiele rules based on this strategyproofness notion.

1 Introduction

A central problem in multi-agent systems is collective decision making: given the preferences of multiple agents over a set of alternatives, a joint decision has to be made. While classic literature for this problem focuses on the case of choosing a single alternative as the winner, there is also a wide range of scenarios where a whole set of winners needs to be elected. For instance, this is the case in parliamentary elections, where the seats of a parliament are assigned to parties based on the voters' preferences. In the literature, parliamentary elections are studied under the term *apportionment* and a crucial assumption for their analysis is that voters are only allowed to vote for a single party (Balinski and Young 2001; Pukelsheim 2014). However, this assumption has recently been criticized because of its lack of flexibility and expressiveness (Brill, Laslier, and Skowron 2018; Brill et al. 2020). Following Brill et al. (2020), we thus study *party-approval elections*. In this setting, the parliament, or more generally a multiset of fixed size, is elected based on the approval ballots of the voters, i.e., each voter reports a set of approved parties instead of only her most preferred one.

Two central desiderata for party-approval elections are *proportional representation* and *strategyproofness*. The former requires that the chosen committee should proportionally reflect the voters' preferences. The latter postulates that no voter can benefit by misreporting her preferences. While Brill et al. (2020) have shown that even core-stable committees, which satisfy one of the highest degrees of proportionality, always exist in party-approval elections, strategyproofness is not yet well-understood for this setting. We thus analyze the trade-off between strategyproofness and proportional representation for party-approval elections in this paper.

Our research question also draws motivation from related models (see Figure 1 for details). Firstly, party-approval elections can be seen as a special case of *approval-based committee (ABC) elections*, where voters approve individual candidates instead of parties and the outcome is a subset of the candidates instead of a multiset. For ABC elections, proportionality and strategyproofness have received significant attention (see, e.g., the survey by Lackner and Skowron (2022)). Unfortunately, these axioms are jointly incompatible for ABC voting rules (Peters 2018) and our study can thus be seen as an attempt to circumvent this impossibility. Even more, there are hints that these axioms could be compatible for party-approval elections: this setting lies logically between ABC elections on the one side, and either *apportionment* (where voters can only approve a single party instead of multiple ones (Balinski and Young 2001; Pukelsheim 2014)) or *fair mixing* (where the outcome is a probability distribution over the parties instead of a multiset (Bogomolnaia, Moulin, and Stong 2005; Aziz, Bogomolnaia, and Moulin 2019)) on the other side. Since strategyproofness and proportionality are compatible in the latter two models, it seems reasonable to conjecture positive results for party-approval elections.

Our contribution. Unfortunately, it turns out that strategyproofness conflicts even with minimal notions of proportional representation in party-approval elections. To prove this, we introduce the notions of weak representation and weak proportional representation, which require that a party is assigned at least 1 (resp. ℓ) out of k available seats if it is uniquely approved by at least an $\frac{1}{k}$ (resp. $\frac{\ell}{k}$) fraction of the voters. Then, we show in Section 3 the following impossibility theorems (k, m, and n denote the numbers of seats, parties, and voters, respectively):

- No anonymous party-approval rule satisfies both strategyproofness and weak representation if $k \geq 3$, $m \geq k+1$, and $2k$ divides n.
- No anonymous party-approval rule satisfies both strategyproofness and weak proportional representation if $k \geq 3$, $m \geq 4$, and $2k$ divides n.

The first result shows that the incompatibility of strategyproofness and proportional representation first observed for ABC elections also prevails for party-approval elections. Even more, our result implies such an impossibility for ABC elections as our setting is more general. The main drawback of the first result is that it requires more parties than seats in the committee. While this assumption is true for many applications inspired from ABC voting, this is not the case in our initial example of parliamentary elections. However, our second impossibility shows that strategyproofness is still in conflict with proportional representation if $k > m$.

We prove both of these results with a computer-aided approach based on SAT solving, which has recently led to a number of sweeping impossibility results (e.g., Brandl et al. 2021; Brandt, Saile, and Stricker 2022). In particular, our computer proof relies on 635 profiles, which makes it the largest computer proof in social choice theory (the previous record is due to Brandl et al. (2021) and uses 386 profiles).

Finally, to derive more positive results, we investigate in Section 4 a weakening of strategyproofness that requires that voters who do not approve any party in the elected committee cannot manipulate. Perhaps surprisingly, many commonly studied committee voting rules fail this condition. On the other hand, we characterize Chamberlin–Courant approval voting as the only Thiele rule satisfying this strategyproofness notion and weak representation, thus proving an attractive escape route to our impossibility results.

Related work. Party-approval elections have been introduced by Brill et al. (2020) who showed that strong proportionality axioms can be satisfied in this setting, but we are not aware of any follow-up paper on this topic yet. We thus draw much inspiration from ABC elections for which there is a large amount of work on proportional representation (e.g., Aziz et al. 2017; Sánchez-Fernández et al. 2017; Peters and Skowron 2020; Brill et al. 2022) and strategyproofness (e.g., Aziz et al. 2015; Peters 2018; Kluiving et al. 2020; Lackner and Skowron 2018; Botan 2021). For instance, Aziz et al. (2017) analyze ABC voting rules with respect to more restrictive variants of weak representation. The main message from work on proportional representation is that there are few ABC voting rules that guarantee strong representation axioms. The results on strategyproofness are mostly negative: after early results (Aziz et al. 2015; Lackner and Skowron 2018) proving that no known rule satisfies both strategyproofness and proportional representation, Peters (2018) showed that these axioms are inherently incompatible for ABC voting rules (see also (Duddy 2014; Kluiving et al. 2020) for related results). Our first impossibility is closely connected to this result but logically independent: while we need stronger strategyproofness and representation axioms and additionally anonymity, our setting is more flexible and we use no efficiency condition.

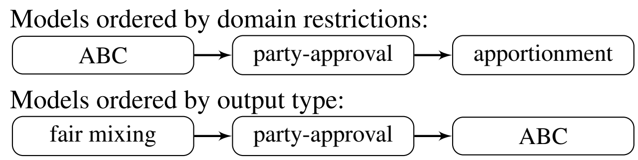

Figure 1: Relation of party-approval elections to other voting settings. An arrow from X to Y means that model X is more general than model Y. In the settings in the top row, elections return sets of alternatives but the models impose different restrictions on the input profiles: for ABC voting every input profile is allowed, for party-approval profiles each voter can for each party (viewed as a set of alternatives) either approve all of its members or none, and for apportionment each voter must approve all members of exactly one party (see Brill et al. (2020) for more details). In the bottom row, the models are ordered with respect to their output type: all of fair mixing, party-approval elections, and ABC elections can take arbitrary approval profiles as input, but fair mixing rules return a probability distribution over the alternatives, party-approval rules choose a multiset of the alternatives, and ABC rules choose a subset of the alternatives. In particular, this shows that party-approval elections can be seen both as generalization and special case of ABC elections.

2 Preliminaries

Let $N = \{1, \dots, n\}$ denote a set of n voters and $\mathcal{P} = \{a, b, c, \dots\}$ a set of m parties. Each voter $i \in N$ is assumed to have a *dichotomous preference relation* over the parties, i.e., she partitions the parties into approved and disapproved ones. The *approval ballot* $A_i \subseteq \mathcal{P}$ of a voter i is the non-empty set of her approved parties. With slight abuse of notation, we omit commas and brackets when writing approval ballots. Let $\mathcal{A}$ denote the set of all possible approval ballots. An *approval profile* $A \in \mathcal{A}^n$ is the collection of the approval ballots of all voters. Given an approval profile A, the goal in *party-approval elections* is to assign a fixed number of seats to the parties. We call such an outcome a *committee*, which is formally a multiset of parties $W : \mathcal{P} \to \mathbb{N}$, and $W(x)$ denotes the number of seats assigned to party x. We extend this notation also to sets of parties $X \subseteq \mathcal{P}$ by defining $W(X) = \sum_{x \in X} W(x)$. Furthermore, we indicate specific committees by square brackets, e.g., $[a, a, b]$ is the committee containing party a twice and party b once. Let $\mathcal{W}_k$ denote the set of all committees of size k.

A *party-approval rule* is a function f which takes an approval profile $A \in \mathcal{A}^n$ and a target committee size k as input and returns a winning committee $W \in \mathcal{W}_k$. In particular, party-approval rules are resolute, i.e., there is always a single winning committee. We define $f(A, k, x)$ as the number of seats assigned to party x by f for the profile A when choosing a committee of size k. Just as for committees, we extend this notion also to sets by defining $f(A, k, X) = \sum_{x \in X} f(A, k, x)$.

Two well-known properties of voting rules are anonymity and Pareto optimality. Intuitively, anonymity requires that all voters are treated equally, i.e., a party-approval rule f is *anonymous* if $f(A,k) = f(A',k)$ for all committee sizes $k \in \mathbb{N}$ and all approval profiles $A, A' \in \mathcal{A}^n$ such that there is a permutation $\pi : N \to N$ with $A'_i = A_{\pi(i)}$.

Next, we say that a party x *Pareto dominates* another party y in an approval profile A if $y \in A_i$ implies $x \in A_i$ for all $i \in N$ and there is a voter $i \in N$ with $x \in A_i$ and $y \notin A_i$. Then, a party-approval rule f is *Pareto optimal* if $f(A,k,y) = 0$ for all approval profiles A, committee sizes k, and parties y that are Pareto dominated in A.

2.1 Proportional Representation

One of the central desiderata in committee elections is to choose a committee that proportionally represents the voters' preferences. The notion of *justified representation*, introduced by Aziz et al. (2017), formalizes this idea by requiring that in a committee of size k, any group of voters $G \subseteq N$ with $|G| \geq \frac{n}{k}$ that agrees on a party should be represented. While this is already a rather weak representation axiom, in this paper we will consider a further weakening which we call *weak representation*. Intuitively, weak representation weakens justified representation by only considering cases where all voters in G uniquely approve a single party x.

Definition 1 (Weak Representation)**.** A party-approval rule f satisfies *weak representation* if $f(A,k,x) \geq 1$ for every profile A, committee size k, party x, and group of voters G such that $|G| \geq \frac{n}{k}$ and $A_i = \{x\}$ for all $i \in G$.

Weak representation can be easily satisfied if we have more seats in the committee than parties by simply assigning at least one seat to every party. This, however, clearly contradicts the idea of proportional representation because a large part of the chosen committee is independent of the voters' preferences. To address this issue, we consider weak proportional representation, which is a weakening of *proportional justified representation* (Sánchez-Fernández et al. 2017). Clearly, weak proportional representation implies weak representation.

Definition 2 (Weak Proportional Representation)**.** A party-approval rule f satisfies *weak proportional representation* if $f(A,k,x) \geq \ell$ for every profile A, committee size k, party x, and group of voters G such that $|G| \geq \ell\frac{n}{k}$ and $A_i = \{x\}$ for all $i \in G$.

2.2 Strategyproofness

Intuitively, strategyproofness requires that a voter cannot benefit by lying about her true preferences. Consequently, if a party-approval rule fails strategyproofness, we cannot expect the voters to submit their true preferences, which may lead to socially undesirable outcomes.

Definition 3 (Strategyproofness)**.** A party-approval rule f is *strategyproof* if $f(A,k,A_i) \geq f(A',k,A_i)$ for all approval profiles A, A', committee sizes k, and voters $i \in N$ such that $A_j = A'_j$ for all $j \in N \setminus \{i\}$.

The motivation for this strategyproofness notion stems from the assumption that voters are indifferent between their approved parties. Then, only the number of seats assigned to these parties matters to the voters. Also, note that this strategyproofness notion is commonly used in ABC voting (e.g., Lackner and Skowron 2018; Botan 2021), often under the name cardinality strategyproofness, and equivalent notions are used for fair mixing (e.g., Bogomolnaia, Moulin, and Stong 2005; Aziz, Bogomolnaia, and Moulin 2019).

Since we will show that strategyproofness is in conflict with minimal representation axioms, we also consider the following weakening which requires that only voters without representation in the committee cannot manipulate.

Definition 4 (Strategyproofness for Unrepresented Voters)**.** A party-approval rule f is *strategyproof for unrepresented voters* if $f(A,k,A_i) \geq f(A',k,A_i)$ for all approval profiles A, A', committee sizes k, and voters $i \in N$ such that $A_j = A'_j$ for all $j \in N \setminus \{i\}$ and $f(A,k,A_i) = 0$.

We believe this to be a sensible relaxation of strategyproofness because voters without any representation in the committee are more prone to manipulate. Firstly, voters who do have some representation may be more cautious to manipulate because they fear losing their representation when misstating their preferences. Secondly, the benefit of having additional representation in the committee is less straightforward than that of being represented at all.

2.3 Party-Approval Rules

Finally, we introduce three classes of party-approval rules. Note that even though we define party-approval rules for a fixed numbers of voters n and parties m, all subsequent rules are independent of such details.

Thiele rules. Thiele rules are arguably the most well-studied class of rules in the ABC setting. Introduced by Thiele (1895), a *w-Thiele rule f* is defined by a non-increasing and non-negative vector $w = (w_1, w_2, \dots)$ and chooses for each committee size k the committee $W \in \mathcal{W}_k$ that maximizes the score $s_w(W,A) = \sum_{i \in N} \sum_{j=1}^{W(A_i)} w_j$. Throughout the paper, we assume without loss of generality that $w_1 = 1$. There are many well-known Thiele rules, such as:

- *approval voting (AV)*: $w = (1,1,1,\dots)$,
- *proportional approval voting (PAV)*: $w = (1, \frac{1}{2}, \frac{1}{3}, \dots)$,
- *Chamberlin–Courant approval voting (CCAV)*: $w = (1,0,0,\dots)$.

Sequential Thiele rules. Sequential Thiele rules are closely related to Thiele rules: instead of optimizing the score of the committee, these rules proceed in rounds and greedily choose in each iteration the party that increases the score of the committee the most. An important example of sequential Thiele rules is *sequential proportional approval voting (seqPAV)* (defined by $w = (1, \frac{1}{2}, \frac{1}{3}, \dots)$) .

Divisor methods based on majoritarian portioning. Brill et al. (2020) introduced the concept of composite party-approval rules, which combine a *portioning method* with an *apportionment method*. In this paper, we focus on an important subclass of such composite rules, namely *divisor methods based on majoritarian portioning*, because many of these rules satisfy strong representation axioms (Brill et al. 2020). These methods first apply *majoritarian portioning* to compute a weight w_x for each party x. Majoritarian portioning works in rounds and in each round, we determine the party x that is approved by the most voters. Then, we set its weight w_x to the number of voters who approve x and remove all corresponding voters from the profile. This process is repeated until no voters are left. Finally, for all parties x that have no weight after all voters were removed, we set $w_x = 0$. After the portioning, we use a *divisor method* to allocate the seats to the parties based on the weights w_x. Divisor methods are defined by a monotone function $g : \mathbb{N}_0 \to \mathbb{R}_{>0}$ and proceed in rounds: in the i-th round, the next seat is assigned to the party x that maximizes $\frac{w_x}{g(t^x_{i-1})}$, where t^x_{i-1} is the number of seats allocated to x in the previous $i-1$ rounds. An example of a divisor method is Jefferson's method (where $g(x) = x + 1$).

Note that all party-approval rules defined above are in principle irresolute, i.e., they may declare multiple committees as tied winners of an election. Since we investigate resolute voting rules in this paper, we assume that ties are always broken lexicographically. Formally, this means that, for every $k \in \mathbb{N}$, there is a linear tie-breaking order $\succ_k$ on the committees $W \in \mathcal{W}_k$ and, if a party-approval rule f declares multiple committees as tied winners, we choose the best one according to $\succ_k$. Similarly, if any rule is tied between multiple parties in a step, the tie is broken according to $\succ_1$. The assumption of lexicographic tie-breaking is standard in the literature on strategyproofness (e.g., Faliszewski, Hemaspaandra, and Hemaspaandra 2010; Aziz et al. 2015).

3 Impossibility Results

In this section, we discuss the incompatibility of strategyproofness and proportional representation for party-approval rules by proving two sweeping impossibility theorems.

Theorem 1. *No party-approval rule simultaneously satisfies anonymity, weak representation, and strategyproofness if* $k \geq 3$*,* $m \geq k + 1$*, and* $2k$ *divides* n*.*

Note that Theorem 1 does not hold for all combinations of k, m, and n: we require that $2k$ divides n and that $m \geq k + 1$. The first assumption is mainly a technical one as we—just like other authors (Peters 2018; Kluiving et al. 2020)—could not find an argument to generalize the impossibility to arbitrary values of n. However, many party-approval rules (e.g., all Thiele rules and sequential Thiele rules) do not change their outcome when adding voters who approve all parties. For such rules, we can extend Theorem 1 to all $n \geq 2k$ by simply adding voters who approve all parties.

On the other side, the assumption that $m \geq k + 1$ is crucial for Theorem 1: if $m \leq k$, every rule that constantly returns a fixed committee W with $W(x) \geq 1$ for all $x \in \mathcal{P}$ satisfies the considered axioms. Nevertheless, we can restore the impossibility by strengthening weak representation to weak proportional representation.

Theorem 2. *No party-approval rule simultaneously satisfies anonymity, weak proportional representation, and strategyproofness if* $k \geq 3$*,* $m \geq 4$*, and* $2k$ *divides* n*.*

We believe that also the proofs of our results are of interest: for showing Theorems 1 and 2, we rely on a computer-aided approach called SAT solving. In the realm of social choice, this technique was pioneered by Tang and Lin (2009) and has by now been used to prove a wide variety of results (e.g., Peters 2018; Endriss 2020; Brandl et al. 2021). We refer to Geist and Peters (2017) for an overview of this technique.

To apply SAT solving to our problems, we proceed in three steps: first, we encode the problem of finding an anonymous party-approval rule that satisfies strategyproofness and weak representation for committees of size $k = 3$, $m = 4$ parties, and $n = 6$ voters as logical formula. By letting a computer program, a so-called SAT solver, show the formula unsatisfiable, we prove the base case of Theorems 1 and 2 for the given parameters. Next, we generalize the impossibility to larger values of k, m, and n based on inductive arguments. Finally, we verify the computer proof. The following subsections discuss each of these steps in detail.

Remark 1. AV satisfies all axioms of Theorem 1 except weak representation, and CCAV satisfies all axioms except strategyproofness. These examples show that these axioms are required for the impossibility. On the other hand, we could not show that anonymity is necessary for the impossibility and we conjecture that this axiom can be omitted.

Remark 2. For electorates where the committee size k is a multiple of the number of voters n, there are voting rules that satisfy weak proportional representation, anonymity, and strategyproofness. We can simply let every voter choose $\frac{k}{n}$ parties of the committee independently of the ballots of other voters. This is an important difference to the impossibility by Peters (2018), which also holds in the case that $n = k$.

Remark 3. If $k = 2$, a variant of AV satisfies all axioms of Theorems 1 and 2. For introducing this rule f, let $\succ$ denote a linear order over the parties and $mAV(A)$ the maximal approval score of a party in the profile A. As first step, f removes clones according to $\succ$, i.e., for all parties x, y such that $x \in A_i$ if and only if $y \in A_i$ for all $i \in N$ and $x \succ y$, we remove y from A. This results in a reduced profile A'. Now, if $mAV(A') \neq \frac{n}{2}$ or there is only a single party with approval score of $\frac{n}{2}$, f assigns both seats to the approval winner. Else, f assigns the seats to the best and second best party with respect to $\succ$ that have an approval score of $\frac{n}{2}$.

3.1 Computer-Aided Theorem Proving

The core observation for computer-aided theorem proving is that for a fixed committee size k and fixed numbers of parties m and voters n, there is a very large but finite number of party-approval rules. Hence, we could, at least theoretically, enumerate all rules and check whether they satisfy our requirements. However, the search space grows extremely fast (for $k = 3$, $m = 4$, and $n = 6$, there are roughly 6.2×10^{14819544} party-approval rules) and we thus use a different idea: we construct a logical formula which is satisfiable if and only if there is an anonymous party-approval rule that satisfies weak representation and strategyproofness for the given parameters of k, m, and n. By showing that the formula is unsatisfiable, we prove Theorems 1 and 2 for fixed parameters. Moreover, we can use computer programs, so-called SAT solvers, to show this.

Subsequently, we explain how we construct the formula and first specify the variables: the idea is to introduce a variable $x_{A,W}$ for each profile $A \in \mathcal{A}^n$ and committee $W \in \mathcal{W}_k$, with the interpretation that $x_{A,W}$ is true if and only if $f(A,k) = W$. However, for this formulation, the mere number of profiles becomes prohibitive when $k = 3$, $m = 4$, and $n = 6$ and we thus apply several optimizations. First, we use anonymity to drastically reduce the number of considered profiles. This axiom states that the order of the voters does not matter for the outcomes and we thus view approval profiles from now on as multisets of approval ballots instead of ordered tuples. Next, we exclude certain approval profiles from the domain by imposing three conditions: *(i)* no voter is allowed to approve all parties, *(ii)* no party can be approved by more than four voters, and *(iii)* the total number of approvals given by all voters does not exceed eleven. We call the domain of all anonymous profiles that satisfy these conditions $\mathcal{A}^n_{SAT}$. Clearly, if there is no anonymous party-approval rule satisfying strategyproofness and weak representation on $\mathcal{A}^n_{SAT}$, there is also no such function on the full domain $\mathcal{A}^n$. For our last optimization, we note that weak representation requires that a committee W cannot be returned for a profile A if there is a party x with $W(x) = 0$ that is uniquely approved by $\frac{n}{k}$ or more voters. Hence, all corresponding variables $x_{A,W}$ must be set to false and we can equivalently omit them. To formalize this, we define $WR(A,k)$ as the set of committees of size k that satisfy weak representation for the profile A. Then, we add for every profile $A \in \mathcal{A}^n_{SAT}$ and every committee $W \in WR(A,k)$ a variable $x_{A,W}$.

Next, we turn to the constraints of our formula. First, we specify that the formula encodes a function f on $\mathcal{A}^n_{SAT}$, i.e., for every profile $A \in \mathcal{A}^n_{SAT}$, there is exactly one committee $W \in WR(A,k)$ such that $x_{A,W} = 1$. For this, we add two types of clauses for every profile A: the first one specifies that at least one committee is chosen for A and the second one that no more than one committee can be chosen.

$$\bigvee_{W \in WR(A,k)} x_{A,W} \qquad \forall A \in \mathcal{A}^n_{SAT}$$

$$\bigwedge_{V,W \in WR(A,k): V \neq W} \neg x_{A,V} \vee \neg x_{A,W} \qquad \forall A \in \mathcal{A}^n_{SAT}$$

Since weak representation and anonymity are encoded in the choice of variables, we only need to add the subsequent constraints for strategyproofness. Here, $A^{A_i \to A_j}$ is the profile derived from A by changing a ballot A_i to A_j.

$$\neg x_{A,V} \vee \neg x_{A',W} \quad \forall A, A' \in \mathcal{A}^n_{SAT}, V \in WR(A,k), W \in WR(A',k) : \exists A_i, A_j \in \mathcal{A} : A' = A^{A_i \to A_j} \wedge W(A_i) > V(A_i)$$

For committees of size $k = 3$, $m = 4$ parties, and $n = 6$ voters, this construction results in a formula containing $21,418,593$ constraints and a state-of-the-art SAT solver, such as Glucose (Audemard and Simon 2018), needs less than a minute to prove its unsatisfiability. Our code also provides options which further reduce the size of the formula to speed up the SAT solving (see the supplementary material for details). Consequently, we derive the following result.

Proposition 1. *There is no party-approval rule that satisfies anonymity, weak representation, and strategyproofness if $k = 3$, $m = 4$, and $n = 6$.*

3.2 Inductive Arguments

Since weak propositional representation implies weak representation, Proposition 1 proves Theorems 1 and 2 for fixed parameters k, m, and n. To complete the proofs of these theorems, we use inductive arguments to generalize the impossibilities to larger parameters and subsequently present them for Theorem 1. For Theorem 2, only the third claim needs to be adapted and the details can be found in the supplementary material.

Lemma 1. *Assume there is no anonymous party-approval rule f that satisfies weak representation and strategyproofness for committees of size k, m parties, and n voters. The following claims hold:*

(1) For every $\ell \in \mathbb{N}$, there is no such rule for committees of size k, m parties, and $\ell \cdot n$ voters.

(2) There is no such rule for committees of size k, $m + 1$ parties, and n voters.

(3) If k divides n, there is no such rule for committees of size $k + 1$, $m + 1$ parties, and $\frac{n(k+1)}{k}$ voters.

Proof sketch. For all three claims, we prove the contrapositive: if there is an anonymous party-approval rule f that satisfies strategyproofness and weak representation for the increased parameters, there is also such a rule g for committees of size k, m parties, and n voters. Subsequently, we discuss how to define the rule g for the three different cases:

(1) Assume there is $\ell \in \mathbb{N}$ such that f is defined for committees of size k, m parties, and $\ell \cdot n$ voters. Given a profile A for m parties and n voters, g copies every voter ℓ times to derive the profile A'. Then, $g(A,k) = f(A',k)$.

(2) Assume f is defined for committees of size k, $m + 1$ parties, and n voters. Given a profile A for m parties and n voters, g first constructs the profile A^{xy} by cloning a party $x \in \mathcal{P}$ into a new party $y \notin \mathcal{P}$. More formally, A^{xy} is defined by A^{xy}_i if $x \notin A_i$ and $A^{xy}_i = A_i \cup \{y\}$ otherwise. Finally, $g(A,k,z) = f(A^{xy},k,z)$ for all $z \neq x$ and $g(A,k,x) = f(A^{xy},k,xy)$.

(3) Assume k divides n and f is defined for committees of size $k + 1$, $m + 1$ parties, and $\frac{n(k+1)}{k}$ voters. In this case, g maps a profile A for m parties and n voters to the profile $\bar{A}^{xy}$ defined as follows: first g derives A^{xy} as explained in the previous case and then it adds $\frac{n}{k}$ voters with ballot xy. Finally, $g(A,k,z) = f(\bar{A}^{xy}, k+1, z)$ for all $z \neq x$ and $g(A,k,x) = f(\bar{A}^{xy}, k+1, xy) - 1$.

For all cases, it remains to show that g is a well-defined party-approval rule that satisfies anonymity, weak representation, and strategyproofness. Due to space restrictions, we explain this only for case (1) and defer the remaining cases to the supplementary material. In this case, g inherits anonymity from f since permuting the voters in A only permutes the voters in the enlarged profile A'. Also, g satisfies weak representation: if $\frac{n}{k}$ or more voters uniquely approve a party x in a profile A, at least $\frac{\ell \cdot n}{k}$ voters uniquely approve x in A'. Thus, $g(A,x) = f(A',x) \geq 1$ because f satisfies weak representation. Finally, we prove that g is strategyproof. Note for this that $f(\bar{A},k,\bar{A}_i) \geq f(\bar{A}',k,\bar{A}_i)$ for all profiles $\bar{A}$, $\bar{A}'$ that only differ in the ballots of voters who report $\bar{A}_i$ in

$\bar{A}$. This is true because we can transform $\bar{A}$ into $\bar{A}'$ by letting voters with ballot $\bar{A}_i$ manipulate one after another, and strategyproofness shows for every step that the number of seats assigned to parties in $\bar{A}_i$ cannot increase. Hence, g is strategyproof because if A and A' only differ in a single ballot A_i, the enlarged profiles $\bar{A}$ and $\bar{A}'$ differ in ℓ voters with ballot A_i. Thus, g meets all requirements in case (1). □

3.3 Verification

Since Proposition 1 is proved by automated SAT solving, there is no complete human-readable proof for verifying Theorems 1 and 2. The standard approach for adressing this issue is to analyze minimal unsatisfiable subsets (MUSes) of the original formula, i.e., subsets of the formula which are unsatisfiable but removing a single constraint makes them satisfiable. Such MUSes are typically much smaller than the original formula, which makes it possible to translate them into a human-readable proof. Unfortunately, this technique does not work for Proposition 1 because all MUSes that we found (by using the programs haifamuc and muser2 (Belov and Marques-Silva 2012; Nadel, Ryvchin, and Strichman 2014)) are huge: even after applying several optimizations, the smallest MUS still contained over 20,000 constraints and 635 profiles. Because of the size of the MUSes, any human-readable proof would be unreasonably long and we thus verify our results by other means. Firstly, we provide the code used for proving Proposition 1 in the supplementary material and will make it publicly available upon acceptance, thus enabling other researchers to reproduce the impossibility.

Secondly, we provide a human-readable proof for a weakening of Proposition 1 that additionally uses Pareto optimality. This proof is derived by applying the computer-aided approach explained in Section 3.1 and by analyzing MUSes of the corresponding formula. Hence, it showcases the correctness of our code. Unfortunately, the proof of this weaker claim still takes 11 pages (even though the used MUSes only consist of roughly 500 constraints), and we thus have to defer it to the supplementary material.

Thirdly, we have—analogous to Brandl et al. (2018) and Brandt, Saile, and Stricker (2022)—verified the correctness of our results with the interactive theorem prover Isabelle/HOL (Nipkow, Paulson, and Wenzel 2002). Such interactive theorem provers support much more expressive logics and we can hence formalize the entire theorems with all the mathematical notions expressed in a similar way as in Section 2. For instance, Figure 2 displays our Isabelle formalization of weak representation. Our Isabelle/HOL implementation thus directly derives Proposition 1 as well as Theorems 1 and 2 from the definitions of the axioms. This releases us from the need to check any intermediate steps encoded in Isabelle because Isabelle checks the correctness of these steps for us. Moreover, Isabelle/HOL is highly trustworthy as all proofs have to pass through an inference kernel, which only supports the most basic logical inference steps. Indeed, experts in automated theorem proving consider Isabelle proofs as more reliable than human-readable proofs (e.g., Hales et al. 2017). Thus, to trust the correctness of our result, one only needs to trust the faithfulness of our Isabelle implementation to the definitions in Section 2.

```
weak_rep_for_anon_papp_rules n 𝒫 k f =
 (anon_PAPP_rule n 𝒫 k f ∧
 (∀A x. anon_papp_profile n 𝒫 A ∧
  k * count A {x} ≥ n → count f(A) x ≥ 1))
```

Figure 2: The Isabelle/HOL code for weak representation. Given the number of voters n, the set of parties $\mathcal{P}$, a target committee size k, and a function f, the code first verifies that f is an anonymous party-approval rule for the given parameters and then requires for every profile A (that is valid for n and $\mathcal{P}$) and every party x that x has at least one seat in $f(A)$ if at least $\frac{n}{k}$ voters uniquely approve x.

4 Strategyproofness for Unrepresented Voters

Since cardinal strategyproofness does not allow for attractive party-approval rules, we consider strategyproofness for unrepresented voters (Definition 4) in this section. Instead of prohibiting all voters from manipulating, this property requires that only voters who do not approve any party in the elected committee cannot manipulate.

As a first result, we prove that CCAV satisfies this axiom and can even be characterized based on strategyproofness for unrepresented voters and weak representation within the class of Thiele rules. Hence, CCAV offers an attractive escape route to Theorem 1 as it satisfies all axioms of the impossibility when weakening strategyproofness.

Theorem 3. *CCAV is the only Thiele rule that satisfies weak representation and strategyproofness for unrepresented voters for all committee sizes k, numbers of parties m, and numbers of voters n.*

Proof. For proving this theorem, we show that CCAV satisfies the given axioms for all k, m, and n (Claim 1), and that no other Thiele rule does so (Claim 2).

Claim 1: We start by proving that CCAV satisfies weak representation and note for this that Aziz et al. (2017) have shown that CCAV satisfies justified representation in the ABC setting. It thus satisfies weak representation for party-approval elections as this axiom is weaker than justified representation and party-approval elections can be seen as special case of ABC elections.

Next, we prove by contradiction that CCAV satisfies strategyproofness for unrepresented voters. Hence, suppose that there are a voter $i \in N$, profiles A^1 and A^2, and a committee size k such that $\mathit{CCAV}(A^2, k, A^1_i) > \mathit{CCAV}(A^1, k, A^1_i) = 0$ and $A^1_j = A^2_j$ for all $j \in N \setminus \{i\}$. To simplify notation, let $W^1 = \mathit{CCAV}(A^1, k)$ and $W^2 = \mathit{CCAV}(A^2, k)$, and define $s(W, A) = |\{i \in N \colon A_i \cap W \neq \emptyset\}|$ as the CCAV-score of a committee W in a profile A. Now, the definition of CCAV requires that $s(W^1, A^1) \geq s(W^2, A^1)$ and $s(W^2, A^2) \geq s(W^1, A^2)$. Moreover, since $W^1(A^1_i) = 0$ and $A^1_j = A^2_j$ for all voters $j \in N \setminus \{i\}$, it follows that $s(W^1, A^2) \geq s(W^1, A^1)$. Finally, we assumed that $W^2(A^1_i) > 0$, which implies that $s(W^2, A^1) \geq s(W^2, A^2)$ since $A^1_j = A^2_j$ for all $j \in N \setminus \{i\}$. By combining these inequalities, we obtain $s(W^2, A^2) \geq s(W^1, A^2) \geq s(W^1, A^1) \geq s(W^2, A^1) \geq s(W^2, A^2)$, which implies that all scores are equal. However, lexicographic tie-breaking implies then that we choose either W^1 or W^2 for both A^1 and A^2, which contradicts that $W^1 = CCAV(A^1, k)$ and $W^2 = CCAV(A^2, k)$.

Claim 2: Next, we show that no other Thiele rule but CCAV satisfies weak representation and strategyproofness for unrepresented voters for all k, m, and n. First, observe that AV clearly fails weak representation. Thus, let f be a w-Thiele rule other than AV and CCAV. We will show that f fails strategy-proofness for unrepresented voters. Note for this that there is an index j with $w_1 > w_j$ since f is not AV. We denote with j_0 the smallest such index, which means that $\forall j < j_0, w_j = w_1 = 1$. If $w_{j_0} = 0$, then $j_0 \geq 3$ because f is not CCAV. Let $\mathcal{P} = \{a_1, \dots, a_{j_0}, b_1, \dots, b_{j_0}\}$ be a set of $m = 2j_0$ parties. We construct the profile A with $n = 2 \cdot \binom{2j_0}{j_0} - 2$ voters and set the target committee size to $k = j_0$. The approval ballots of the voters are defined as follows: voter 1 reports $\{a_1, \dots, a_{j_0}\}$, voter 2 reports $\{b_1\}$ and for every set $X \subseteq \mathcal{P}$ with $|X| = j_0$, $X \neq \{a_1, \dots, a_{j_0}\}$, and $X \neq \{b_1, \dots, b_{j_0}\}$, there are two voters who report X as their ballot.

First, note that every party appears in exactly $n_c = 2\binom{2j_0-1}{j_0-1} - 2$ ballots of the voters $N_c = N \setminus \{1, 2\}$. Consequently, every committee W of size j_0 gets a total of $\sum_{x \in \mathcal{P}} W(x)|\{i \in N_c : x \in A_i\}| = j_0 n_c$ approvals from these voters. We use this fact to compute the scores of a committee W derived from these voters. Observe that the committees $W_A = [a_1, \dots, a_{j_0}]$ and $W_B = [b_1, \dots, b_{j_0}]$ receive a score of $j_0 n_c$ from the voters in N_c because none of them approves all parties in the committee and $w_1 = \dots = w_{j_0-1} = 1$. On the other hand, for every other committee W, there are at least two voters who approve all parties in W. Hence, these voters assign a score of $j_0 - 1 + w_{j_0}$ to the committee. Since the total sum of approvals is constant we derive that the remaining voters in N_c assign at most a score of $j_0(n_c - 2)$ to W. Hence, the score of W among voters in N_c is upper bounded by $j_0 n_c - 2(1 - w_{j_0})$. Finally, if we add the first two voters, W_A obtains a score of $j_0 n_c + j_0 - 1 + w_{j_0}$, W_B of $j_0 n_c + 1 < j_0 n_c + j_0 - 1 + w_{j_0}$ (because either $j_0 \geq 3$ or $j_0 = 2$ and $w_{j_0} > 0$), and the scores of other committees is at most $j_0 n_c - 2(1 - w_{j_0}) + j_0 < j_0 n_c + j_0 - 1 + w_{j_0}$ (since $w_{j_0} < 1$). Hence, $f(A, j_0) = W_A$.

Now, consider the profile A' derived from A by changing the approval ballot of voter 2 to $\{b_1, \dots, b_{j_0}\}$. Then, the score of the committee W_A does not change and the score of W_B is now equal to the score of W_A. Moreover, the same argument as before shows that the score of all other committees is still strictly lower. Hence, committees W_A and W_B are now tied for the win. If the tie-breaking favors W_B over W_A, we thus have $f(A', j_0) = W_B$ and voter 2 can manipulate even though $f(A, j_0, A_2) = 0$. Otherwise, we can exchange the roles of $\{a_1, \dots, a_{j_0}\}$ and $\{b_1, \dots, b_{j_0}\}$. Hence, f fails strategyproofness for unrepresented voters. □

A natural follow-up question to Theorem 3 is whether party-approval rules other than Thiele rules satisfy strategyproofness for unrepresented voters. We partially answer this question by showing that all sequential Thiele rules (but AV) and all divisor methods based on majoritarian portioning (but AV) fail this axiom. Hence, even this weak notion of strategyproofness is a challenging axiom for party-approval elections. We defer the proof of this theorem completely to the supplementary material; it works by constructing counterexamples similar to Claim 2 in Theorem 3.

Theorem 4. *All sequential Thiele rules except AV and all divisor methods based on majoritarian portioning except AV fail strategyproofness for unrepresented voters for some committee size k, number of parties m, and number of voters n.*

Remark 4. CCAV becomes highly indecisive if $k \geq m$ since every voter will approve at least one party in the chosen committee. Thus, many seats of the committee will be assigned by the tie-breaking. Hence, CCAV is no attractive rule if $k > m$. Similar arguments show that all w-Thiele rules that have an index j with $w_j = 0$ are strategyproof for unrepresented voters if $k \geq (j-1)m$: in this case, these rules always choose a committee which guarantees every voter $j - 1$ representatives and strategyproofness for unrepresented voters is trivially satisfied. Consequently, Theorem 3 needs to quantify over the committee size, number of parties, and number of voters.

Remark 5. All results of this section carry over into the ABC setting. For the negative results this follows from the fact that party-approval elections can be seen as a special case of ABC elections (see Figure 1). The first claim of Theorem 3 holds since our proof directly translates into the ABC setting.

5 Conclusion

We study the compatibility of strategyproofness and proportional representation for party-approval multiwinner elections, where a multiset of the parties is chosen based on the voters' approval ballots. First, we prove based on a computer-aided approach that strategyproofness and minimal notions of proportional representation are incompatible for anonymous party-approval rules. Thus, the incompatibility of strategyproofness and proportional representation first observed by Peters (2018) for approval-based committee voting rules (which return sets instead of multisets) also prevails in our more flexible setting. As a second contribution, we investigate a weakening of strategyproofness which requires that only voters who do not approve any member of the committee cannot manipulate. Perhaps surprisingly, almost all commonly studied party-approval rules fail even this very weak strategyproofness notion. Conversely, we can characterize Chamberlin–Courant approval voting as the unique Thiele rule that satisfies strategyproofness for unrepresented voters and a weak representation axiom, thus offering an attractive escape route to our previous impossibility theorem.

Our work offers several directions for future extensions. In particular, we feel that strategyproofness for unrepresented voters deserves more attention; for example, we have to leave it open whether weak proportional representation is compatible with this axiom. Furthermore, one can see strategyproofness and strategyproofness for unrepresented voters as two extreme cases of a parameterization of strategyproofness and it thus might be interesting to consider quantified strategyproofness notions for party-approval elections.

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